

## MANY SYMMETRICALLY INDIVISIBLE STRUCTURES

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ABSTRACT. A structure  $\mathcal{M}$  in a first-order language  $\mathcal{L}$  is *indivisible* if for every coloring of  $M$  in two colors, there is a monochromatic  $\mathcal{M}' \subseteq \mathcal{M}$  such that  $\mathcal{M}' \cong \mathcal{M}$ . Additionally, we say that  $\mathcal{M}$  is *symmetrically indivisible* if  $\mathcal{M}'$  can be chosen to be *symmetrically embedded* in  $\mathcal{M}$  (that is, every automorphism of  $\mathcal{M}'$  can be extended to an automorphism of  $\mathcal{M}$ ). In the following paper we give a general method for constructing new symmetrically indivisible structures out of existing ones. Using this method, we construct  $2^{\aleph_0}$  many non-isomorphic symmetrically indivisible countable structures in given (elementary) classes and answer negatively the following question from [HKO11]: Let  $\mathcal{M}$  be a symmetrically indivisible structure in a language  $\mathcal{L}$ . Let  $\mathcal{L}_0 \subseteq \mathcal{L}$ . Is  $\mathcal{M} \upharpoonright \mathcal{L}_0$  symmetrically indivisible?

## 1. INTRODUCTION

The notion of indivisibility of relational first-order structures and metric spaces is well studied in Ramsey theory ([KR86], [EJS91], [EJS93] are just a few examples of the extensive study in this area). Recall that a structure  $\mathcal{M}$  in a relational first-order language is indivisible, if for every coloring of its universe  $M$  in two colors, there is a monochromatic substructure  $\mathcal{M}' \subseteq \mathcal{M}$  such that  $\mathcal{M}' \cong \mathcal{M}$ . Rado's random graph, the ordered set of natural numbers and the ordered set of rational numbers are just a few of the many examples. Weakenings of this notions have also been studied (see [Sau14]). A known extensively studied strengthening of this notion is the pigeonhole property. A first-order relational structure  $X$  admits the pigeonhole property if whenever  $X$  is the union of two disjoint substructures  $Y$  and  $Z$ , at least one of  $Y$  and  $Z$  is isomorphic to  $X$ . Examples of such structures include the random graph and the random  $n$ -hypergraph, though in general such structures are very rare. (See [Cam10] for further reading.) For an extensive review on the subject — see appendix A in [Fra00].

In [GK11], a notion of symmetrized Ramsey theory was introduced, and in [HKO11] a new strengthening of the notion of indivisibility was investigated: We say that a substructure  $\mathcal{N} \subseteq \mathcal{M}$  is *symmetrically embedded* in  $\mathcal{M}$  if every automorphism of  $\mathcal{N}$  extends to an automorphism of  $\mathcal{M}$ . We say that  $\mathcal{M}$  is *symmetrically indivisible* if for every coloring of  $M$  in two colors, there is a monochromatic  $\mathcal{M}' \subseteq \mathcal{M}$  such that  $\mathcal{M}'$  is isomorphic to  $\mathcal{M}$  and  $\mathcal{M}'$  is symmetrically embedded in  $\mathcal{M}$ .

In [HKO11], several examples of symmetrically indivisible structures have been introduced. Examples include the random graph ([GK11]), the ordered rational numbers, the ordered natural numbers, the universal  $n$ -hypergraph.

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This paper is part of the author's M.Sc. thesis done under the supervision of Dr. Assaf Hasson at The Department of Mathematics, Ben-Gurion University of the Negev.

In Section 2 we will present a method of constructing new symmetrically indivisible structures out of existing ones and using this method, construct  $2^{\aleph_0}$  many non-isomorphic symmetrically indivisible countable linear orders and  $2^{\aleph_0}$  many non-isomorphic symmetrically indivisible countable graphs. We give a sufficient conditions for a class of  $\mathcal{L}$ -structures to have  $2^{\aleph_0}$  many symmetrically indivisible structures. We note that these conditions are met by the class of  $n$ -hypergraphs, graphs edge-colored in  $k \leq \omega$  colors, and trivially by the class of partial orders as a super-class of linear orders and with an aim of finding a general claim.

In Section 3 we make further use of this method to construct an example that answers negatively a question asked in [HKO11]: Let  $\mathcal{M}$  be a symmetrically indivisible structure in a language  $\mathcal{L}$ . Let  $\mathcal{L}_0 \subseteq \mathcal{L}$ . Is  $\mathcal{M} \upharpoonright \mathcal{L}_0$  symmetrically indivisible? It is clear that if  $\mathcal{M}$  is indivisible then  $\mathcal{M} \upharpoonright \mathcal{L}_0$  is indivisible, but for a symmetrically embedded  $\mathcal{M}_0 \subseteq \mathcal{M}$ ,  $\mathcal{M}_0 \upharpoonright \mathcal{L}_0$  is not necessarily symmetrically embedded in  $\mathcal{M}$ , thus this question does not seem to have an immediate answer.

## 2. MANY EXAMPLES FOR SYMMETRICALLY INDIVISIBLE STRUCTURES

In this section we will show how to construct many symmetrically indivisible structures based on existing ones.

**Definition 2.1.** For two first-order structures  $\mathcal{M}, \mathcal{N}$  an embedding  $e : \mathcal{N} \rightarrow \mathcal{M}$  is called *symmetric* if every automorphism of  $e[\mathcal{N}]$  extends to an automorphism of  $\mathcal{M}$ . So a substructure  $\mathcal{N} \subset \mathcal{M}$  is symmetrically embedded if the inclusion map  $\iota$  is symmetric.

**Definition 2.2.** Let  $\mathcal{L}$  be a first-order language and let  $\mathcal{M}, \mathcal{N}$  be  $\mathcal{L}$ -structures. We denote  $\mathcal{M} \lesssim_{(s)} \mathcal{N}$  if there is a (*symmetric*) embedding  $e : \mathcal{M} \hookrightarrow \mathcal{N}$ . Similarly, we denote  $\mathcal{M} \subseteq_{(s)} \mathcal{N}$  if  $\mathcal{M}$  is a (symmetrically embedded) *substructure* of  $\mathcal{N}$ .

Finally, we denote  $\mathcal{M} \sim_{(s)} \mathcal{N}$  if  $\mathcal{M} \lesssim_{(s)} \mathcal{N}$  and  $\mathcal{N} \lesssim_{(s)} \mathcal{M}$ .

### Proposition 2.3.

- (1) If  $\mathcal{M}$  and  $\mathcal{N}$  are  $\mathcal{L}$ -structures such that  $\mathcal{M} \sim \mathcal{N}$  then  $\mathcal{M}$  is indivisible iff  $\mathcal{N}$  is indivisible.
- (2) If  $\mathcal{M}$  and  $\mathcal{N}$  are  $\mathcal{L}$ -structures such that  $\mathcal{M} \sim_s \mathcal{N}$  then  $\mathcal{M}$  is *symmetrically* indivisible iff  $\mathcal{N}$  is *symmetrically* indivisible.

*Proof.*

- (2) Because  $\sim_s$  is an equivalence relation, it is enough to show one direction: suppose  $\mathcal{M}$  is symmetrically indivisible, and let  $c : \mathcal{N} \rightarrow \{\text{red}, \text{blue}\}$  be a coloring of  $\mathcal{N}$ . Since  $\mathcal{M} \lesssim_s \mathcal{N}$ , let  $\mathcal{M}_0 \subseteq_s \mathcal{N}$  such that  $\mathcal{M}_0 \cong \mathcal{M}$ , so  $c \upharpoonright \mathcal{M}_0$  is a coloring of  $\mathcal{M}_0$ , and since  $\mathcal{M}$  is symmetrically indivisible, so is  $\mathcal{M}_0$  and there is a monochromatic  $\mathcal{M}'_0 \subseteq_s \mathcal{M}_0$  such that  $\mathcal{M}'_0$  is isomorphic to  $\mathcal{M}_0 \cong \mathcal{M}$ . Now, since  $\mathcal{N} \lesssim_s \mathcal{M} \cong \mathcal{M}'_0$ , there is  $\mathcal{N}_0 \subseteq_s \mathcal{M}'_0$  such that  $\mathcal{N}_0$  is isomorphic to  $\mathcal{N}$  and since  $\mathcal{M}'_0$  is monochromatic, so is  $\mathcal{N}_0$ . Now, since  $\mathcal{N}_0 \subseteq_s \mathcal{M}' \subseteq_s \mathcal{M} \subseteq_s \mathcal{N}$ , by transitivity  $\mathcal{N}_0 \subseteq_s \mathcal{N}$ .
- (1) Repeat the same argument, omitting "symmetric".

□

Now we are ready to construct  $2^{\aleph_0}$  examples of symmetrically indivisible graphs and  $2^{\aleph_0}$  examples of symmetrically indivisible linear orders, both based on known symmetrically indivisible structures, and the equivalence relation  $\sim_s$ :

Recall:

**Fact 2.4.** the random graph and  $(\mathbb{Q}, <)$  are symmetrically indivisible. ([GK11], [HKO11])

In [Hen71] it was shown that:

**Fact 2.5.** Let  $\Gamma$  be the random graph. For every countable graph  $G$ ,  $G \lesssim_s \Gamma$ .

**Corollary 2.6.** Every countable graph which symmetrically embeds  $\Gamma$  is symmetrically indivisible.

*Proof.* Let  $G$  be a countable graph which symmetrically embeds  $\Gamma$ , then by definition  $G \sim_s \Gamma$  thus by 2.4 combined with Proposition 2.3,  $G$  is symmetrically indivisible.  $\square$

For  $(\mathbb{Q}, <)$  we have a result similar to 2.5:

**Proposition 2.7.** For every countable linear order  $A$ ,  $A \lesssim_s \mathbb{Q}$ .

*Proof.* Let  $A[\mathbb{Q}]$  be the lexicographic order on  $A \times \mathbb{Q}$ . This is a countable dense linear order without end-points (DLO). By  $\aleph_0$ -categoricity of DLO, it is isomorphic to  $(\mathbb{Q}, <)$ .

For a fixed  $q \in \mathbb{Q}$ , the induced substructure on  $A \times \{q\}$  is isomorphic to  $A$  and the fact that it is symmetrically embedded can be easily verified and is actually a special case of Lemma 2.8 of [HKO11].  $\square$

From this we have, exactly like Corollary 2.6 for graphs :

**Corollary 2.8.** Every countable linear order which symmetrically embeds  $(\mathbb{Q}, <)$  is symmetrically indivisible.

**Definition 2.9.** Let  $G, H$  be graphs, for convenience assume  $|G| \cap |H| = \emptyset$ . We define  $G +^{\mathcal{G}} H$  the graph whose universe is  $|G| \cup |H|$  and  $E^{(G +^{\mathcal{G}} H)} := E^G \cup E^H$ .

Namely,  $G +^{\mathcal{G}} H$  is just the "disjoint union of graphs" as known in graph theory and denoted  $G \cup H$ .

**Remark 2.10.** Let  $G, H, K$  be graphs.

- (1)  $G, H \lesssim_s G +^{\mathcal{G}} H$
- (2)  $G +^{\mathcal{G}} H = H +^{\mathcal{G}} G$
- (3)  $(G +^{\mathcal{G}} H) +^{\mathcal{G}} K = G +^{\mathcal{G}} (H +^{\mathcal{G}} K)$
- (4) If  $C \subseteq G$  is a union of connected component, then  $G = (G \setminus C) +^{\mathcal{G}} C$

**Proposition 2.11.** Let  $G, H$  be graphs such that  $\Gamma +^{\mathcal{G}} G \cong \Gamma +^{\mathcal{G}} H$  then  $G \cong H$ .

*Proof.* Let  $\phi : \Gamma +^{\mathcal{G}} G \rightarrow \Gamma +^{\mathcal{G}} H$  be an isomorphism. Since  $\phi$  maps connected components onto connected components and  $\Gamma$  is connected, either  $\phi[\Gamma] = \Gamma$  or  $\phi[\Gamma] \subseteq H$ . In the first case  $\phi[G] = H$  and  $\phi \upharpoonright G : G \rightarrow H$  is an isomorphism, in the second case – by Remark 2.10

$$H +^{\mathcal{G}} \Gamma = (H \setminus \phi[\Gamma]) +^{\mathcal{G}} \phi[\Gamma] +^{\mathcal{G}} \Gamma$$

thus  $\phi \upharpoonright G : G \rightarrow (H \setminus \phi[\Gamma]) +^{\mathcal{G}} \Gamma$  is an isomorphism, but  $(H \setminus \phi[\Gamma]) +^{\mathcal{G}} \Gamma \cong H$ .  $\square$

**Definition 2.12.** Let  $A$  and  $B$  be linear orders, for convenience assume  $|A| \cap |B| = \emptyset$ . We define  $A +^{lo} B$  the linear order whose universe is  $|A| \cup |B|$  and

$$<^{A +^{lo} B} := <^A \cup <^B \cup \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Namely,  $A +^{lo} B$  is the "concatenation" of  $A$  and  $B$  – just putting  $B$  right after  $A$ .

**Proposition 2.13.** Let  $X$  be a linear order such that  $|X| = \{x, y\}$ ,  $<^X = (x, y)$ ,  $A, B$  be linear orders such that  $(\mathbb{Q} +^{lo} X +^{lo} A) \cong (\mathbb{Q} +^{lo} X +^{lo} B)$  then  $A \cong B$ .

*Proof.* Let  $\phi : (\mathbb{Q} +^{lo} X +^{lo} A) \rightarrow (\mathbb{Q} +^{lo} X +^{lo} B)$  be an isomorphism.  $x$  is the minimal element with an immediate successor and  $y$  is the immediate successor of  $x$  in both losets, thus  $\phi(x) = x$ , and  $\phi(y) = y$ . thus

$$\phi[A] = \phi \{ z \in \mathbb{Q} +^{lo} X +^{lo} A \mid y < z \} = \{ z \in \mathbb{Q} +^{lo} X +^{lo} B \mid y < z \} = B$$

□

Similarly to Remark 2.10:

**Remark 2.14.** If  $A, B$  are losets, then  $A, B \lesssim_s (A +^{lo} B)$

**Corollary 2.15.**

- (1) There is a 1-1 map between isomorphism classes of countable graphs and isomorphism classes of countable symmetrically indivisible graphs.
- (2) There is a 1-1 map between isomorphism classes of countable *losets* and isomorphism classes of countable symmetrically indivisible *losets*.

*Proof.* Consider the maps:

- (1)  $G \mapsto \Gamma +^G G$
- (2)  $A \mapsto \mathbb{Q} +^{lo} \{x\} +^{lo} \{y\} +^{lo} A$

By Proposition 2.11 and Proposition 2.13 these are 1-1. By Remark 2.10 and Corollary 2.6, all the graphs of the form  $\Gamma +^G G$  are symmetrically indivisible. By Remark 2.14 and Corollary 2.8 all the losets of the form  $\mathbb{Q} +^{lo} \{x\} +^{lo} \{y\} +^{lo} A$  are symmetrically indivisible. □

We conclude this section with an attempt to generalize both constructions. As mentioned in the introduction, classic examples of symmetrically indivisible structure, in addition to the random graph and the ordered rational numbers, include the universal  $n$ -hypergraph and the universal edge-colored graph in  $k \leq \omega$  many colors defined below:

**Definition 2.16.** For  $k \leq \omega$ , an edge-colored graph in  $k$  many colors  $G$  is a graph whose edges are colored in  $k$  many colors – i.e. it is a structure in the language  $\mathcal{L}_k := \{R_i\}_{i \in k}$  such that  $\{G \upharpoonright R_i\}_{i \in k}$  are edge-disjoint graphs.

For a fixed  $k$ , the class of finite edge-colored graphs in  $k$  many colors is a Fraïssé class. We denote its Fraïssé limit by  $\Gamma_k$ .

In an attempt to generalize Corollary 2.15, we haven't had much success giving an interesting generalization other than the trivial one, that goes as follows:

**Proposition 2.17.** Assume  $\mathcal{C}$  is a class of countable structures in a fixed language  $\mathcal{L}$ , along with a symmetrically indivisible structure  $\mathcal{M} \in \mathcal{C}$  such that for every  $C' \in \mathcal{C}$ ,  $C' \lesssim_s \mathcal{M}$  and a binary operation on  $\mathcal{C}$ ,  $+^C$  satisfying  $\mathcal{M} +^C C_1 \cong \mathcal{M} +^C C_2 \implies C_1 \cong C_2$  and  $A, B \lesssim_s A +^C B$ , then there is a 1-1 map between structures of  $\mathcal{C}$  upto isomorphism and symmetrically indivisible structures of  $\mathcal{C}$  upto isomorphism.

Regarding  $n$ -hypergraphs and edge-colored graphs: A similar construction to that of Theorem 3.1 in [Hen71] can give us a result similar to 2.5 and Proposition 2.7:

- (1) The universal  $n$ -hypergraph symmetrically embeds every  $n$ -hypergraph.

(2)  $\Gamma_k$  symmetrically embeds every edge-colored graph in  $k$  colors.

For  $n$ -hypergraphs and edge-colored graphs,  $+$  will be just the disjoint union, similar to  $+^G$ . Proposition 2.17 gives us a 1-1 map between  $n$ -hypergraphs upto isomorphism and symmetrically indivisible hypergraphs upto isomorphism and the same for edge-colored graphs in  $k \leq \omega$  colors.

### 3. A SYMMETRICALLY INDIVISIBLE STRUCTURE WITH A REDUCT THAT IS NOT SYMMETRICALLY INDIVISIBLE

Recall that in [HKO11] the following question has been asked:

**Question 3.1.** Let  $\mathcal{M}$  be a symmetrically indivisible structure in a language  $\mathcal{L}$ . Let  $\mathcal{L}_0 \subseteq \mathcal{L}$ . Is  $\mathcal{M} \upharpoonright \mathcal{L}_0$  symmetrically indivisible?

In this section, we will construct an example answering this question negatively.

First we construct an indivisible structure which is not symmetrically indivisible. The existence of such a structure is a necessary condition for the existence of an example for 3.1, since if  $\mathcal{M}$  is indivisible (in particular if it is symmetrically indivisible) then  $\mathcal{M} \upharpoonright \mathcal{L}_0$  is also indivisible.

#### 3.1. $\Gamma^*$ – an example of an indivisible structure which is not symmetrically indivisible.

Throughout this subsection  $\Gamma$  will denote the random graph. The indivisibility of the random graphs is a well known fact that dates back to its definition in [Rad64]. An easy proof is given in [Hen71].

**Definition 3.2.** We define the graph  $\Gamma^*$  as follows:

let  $\{K_n\}_{n < \omega}$  be disjoint sets, satisfying  $|K_n| = n$  and let  $\{g_n\}_{n < \omega}$  be an enumeration of  $\Gamma$ . The universe of  $\Gamma^*$  is defined to be  $|\Gamma^*| = |\Gamma| \cup \bigcup_{n < \omega} K_n$  and the edges are defined as follows:

$$E^{\Gamma^*} = E^{\Gamma} \cup \bigcup_{n < \omega} \{ (a, b) \mid a, b \in K_n \cup \{g_n\} \}$$

In words, if  $\{g_n\}_{n \in \omega}$  enumerates the vertices of  $\Gamma$ , for each  $n \in \omega$  we add a clique  $K_n$  and connect it to  $g_n$ .

**Lemma 3.3.**  $\Gamma^*$  is not rigid

*Proof.* for every  $n \geq 2$  and for every two distinct  $a, b \in K_n$ , there is an automorphism of  $\Gamma^*$  swapping  $a$  with  $b$  and fixing all other vertices.  $\square$

**Lemma 3.4.** If  $\sigma : \Gamma^* \rightarrow \Gamma^*$  is an automorphism of  $\Gamma^*$  then  $\sigma \upharpoonright \Gamma = Id_{\Gamma}$

*Proof.* For each  $n \in \omega$ ,  $K_n$  is the set of vertices in  $\Gamma^*$  of degree precisely  $n$ , and  $g_n$  is the unique vertex of infinite degree connected to all vertices in  $K_n$ . Thus  $\sigma(g_n) = g_n$ .  $\square$

**Proposition 3.5.**  $\Gamma^*$  is indivisible but not symmetrically indivisible.

*Proof.* First,  $\Gamma^*$  is indivisible, since clearly  $\Gamma \lesssim \Gamma^*$  and by 2.5,  $G \lesssim_s \Gamma$  for every countable graph  $G$ . In particular  $\Gamma^* \lesssim \Gamma$ , thus  $\Gamma^* \sim \Gamma$  and we have  $\Gamma$  is indivisible thus, by Proposition 2.3,  $\Gamma^*$  is indivisible.

To show  $\Gamma^*$  is not symmetrically indivisible, let  $c : \Gamma^* \rightarrow \{red, blue\}$  a colouring of  $\Gamma^*$  defined by

$$c(x) = \begin{cases} red & \text{if } x \in \Gamma \\ blue & \text{if } x \notin \Gamma \end{cases}$$

Since there is no blue vertex of infinite degree there is no blue copy of  $\Gamma$ , and therefore also  $\Gamma^*$  does not embed in the blue sub-graph. Let  $\Gamma^{*'} be a red substructure isomorphic to  $\Gamma^*$ . By lemma 3.3 and Lemma 3.4  $\Gamma^{*'}$  has an automorphism that cannot be extended to an automorphism of  $\Gamma^*$ , namely  $\Gamma^{*'}$  is not symmetrically embedded in  $\Gamma^*$ .  $\square$$

### 3.2. Enumeration endowments.

Throughout this subsection – fix an ultrahomogeneous structure  $U$  in a relational language  $\mathcal{L}$  satisfying the pigeonhole principal and a structure  $U^*$  not symmetrically indivisible such that  $U \lesssim U^* \lesssim U$ . (For example, the random graph  $\Gamma$  and  $\Gamma^*$  defined above.)

Before we continue our construction, we give a general claim about the pigeonhole property:

**Lemma 3.6.** If  $a_1, \dots, a_n, b \in U$  are distinct and  $g : \{a_1, \dots, a_n\} \rightarrow U$  is a partial isomorphism, then the substructure whose universe is

$$S := \{x \in U \mid g \cup \langle b, x \rangle \text{ is a partial isomorphism}\}$$

is isomorphic to  $U$ , in particular,  $S$  is infinite.

*Proof.* By ultrahomogeneity,  $S$  is non-empty and since  $a_1, \dots, a_n, b$  are distinct,  $g(a_1), \dots, g(a_n) \notin S$ . Let  $s \in S$  and let  $\hat{g} = g \cup \{b, s\}$ . By the pigeonhole property, either  $S$  or  $U \setminus S$  is isomorphic to  $U$ . Assume towards a contradiction that there is an isomorphism  $\phi : U \rightarrow U \setminus S$ .

$$\phi^{-1} \upharpoonright \{\phi \circ g(a_1), \dots, \phi \circ g(a_n)\}$$

is a partial isomorphism of  $U \setminus S$ , and thus by ultrahomogeneity, there is a  $y \in U \setminus S$  such that

$$f := (\phi^{-1} \upharpoonright \{\phi \circ g(a_1), \dots, \phi \circ g(a_n)\}) \cup \{\langle \phi(s), y \rangle\}$$

is a partial isomorphism. Now  $f \circ \phi \circ \hat{g}$  is a partial isomorphism extending  $g$  and  $f \circ \phi \circ \hat{g}(b) \in U \setminus S$ , contradicting the definition of  $S$ .  $\square$

**Definition 3.7.** For an  $\mathcal{L}$  structure  $\mathcal{M}$ , we say an expansion of  $\mathcal{M}$  to  $\mathcal{L} \cup \{<\}$  is an enumeration endowment if  $<$  is of order type  $\omega$ .

Note that two enumeration endowments of the same structure, even in the ultrahomogeneous context, are not necessarily isomorphic.

**Definition 3.8.** Recall that for a first-order relational structure  $\mathcal{M}$ ,  $\text{age}(\mathcal{M})$  is the class of all finite structures which are embeddable in  $\mathcal{M}$ .

**Lemma 3.9.** Let  $A$  be a countable structure with  $\text{age}(A) \subseteq \text{age}(U)$ . If  $U^{<}, A^{<}$  are enumeration endowments of  $U, A$  respectively, then  $A^{<}$  embeds into  $U^{<}$  (as  $\mathcal{L} \cup \{<\}$ -structures).

In particular,  $U^{<}$  is indivisible.

*Proof.* Let  $\langle u_i : i \in \omega \rangle, \langle a_i : i \in \omega \rangle$  be enumerations of  $U$  and  $A$  respectively, compatible with the given enumeration endowments.

We construct the embedding inductively:

- Since  $\text{age}(A) \subseteq \text{age}(U)$ , there is a  $u \in U$  such that  $\langle a_0, u \rangle$  is a partial isomorphism. Let  $e_0 := \langle a_0, u \rangle$  for such a  $u$ .
- By Lemma 3.6, for every  $i \in \omega$ ,

$$S := \{ x \in U \mid e_i \cup \langle a_{i+1}, x \rangle \text{ is a partial } \mathcal{L}\text{-isomorphism} \}$$

is infinite. choose  $u \in S$  such that  $e(a_i) < u$  in the enumeration endowment and let  $e_{i+1} = e_i \cup \langle a_{i+1}, u \rangle$ .

Let

$$e := \bigcup_{i \in \omega} e_i.$$

By the construction  $e$  is an ascending union of  $\mathcal{L}$ -partial isomorphisms, so it is an  $\mathcal{L}$ -embedding. Furthermore it is order preserving – thus it is an  $\mathcal{L} \cup \{<\}$ -embedding.

Now to show  $U^{<}$  is indivisible, let  $c : U \rightarrow \{\text{red}, \text{blue}\}$ . By indivisibility of  $U$  as an  $\mathcal{L}$ -structure, there is a monochromatic  $U'$   $\mathcal{L}$ -isomorphic to  $U$ . As an induced  $\mathcal{L} \cup \{<\}$ -structure,  $U'$  is an enumeration endowment of  $U$  (not necessarily isomorphic to  $U^{<}$ ). By the present lemma,  $U^{<}$  embeds into  $U'$  and it is monochromatic.  $\square$

**Remark 3.10.** Note that since  $\langle \omega, < \rangle$  is rigid, every enumeration endowment is rigid as well, thus in the context of enumerated graphs, symmetric indivisibility and indivisibility coincide.

**Theorem 3.11.** A reduct of a symmetrically indivisible structure to a sub-language is not necessarily symmetrically indivisible.

*Proof.* Consider  $U$  and  $U^*$  above. Assume for simplicity  $U \subseteq U^*$ . Let  $(U^*)^{<}$  be an enumeration endowment of  $U^*$  and let  $U^{<}$  be the induced  $\mathcal{L} \cup \{<\}$ -substructure on  $U$ . Notice that  $U^{<}$  is an enumeration endowment of  $U$  and thus by Lemma 3.9,

$$(U^*)^{<} \lesssim U^{<}$$

so by rigidity,

$$(U^*)^{<} \lesssim_s U^{<} \lesssim_s (U^*)^{<}$$

By Lemma 3.9 and by rigidity  $U^{<}$  is symmetrically indivisible and thus by Proposition 2.3, so is  $(U^*)^{<}$ . But  $U^{<} \upharpoonright \mathcal{L} = U^*$  is not symmetrically indivisible.  $\square$

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